Relative Extrema

Aim: Find maximum / minimum values of a function.

c must be in domain

in some open interval containing c

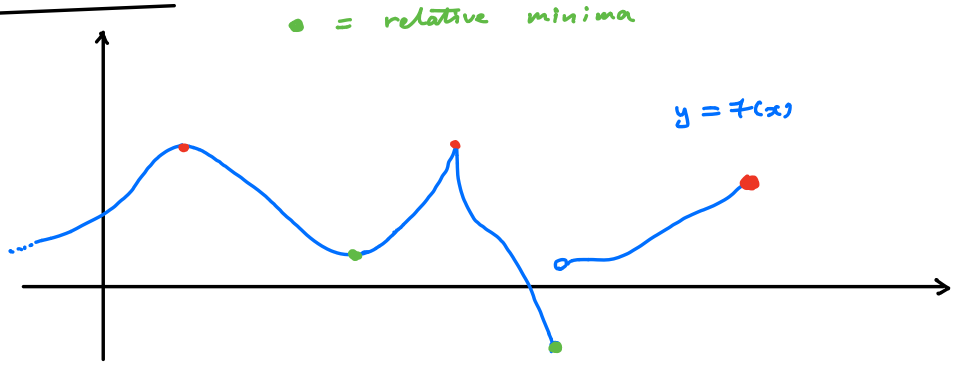
Relative Max: $f(c)$ such that $f(x) \le f(c)$ for all x near c

Relative Min: $f(c)$ such that $f(x) \ge f(c)$ for all x near c

Called relative extrema

Remark: It's possible c is at an endpoint of an interval.

Basic Picture:
● = relative maxima
● = relative minima



Important Fact:

$f(c)$ a relative max / min $\Rightarrow$ c is a critical number of f
($f(c)$ exists and
A/ $f'(c) = 0$ or B/ f' discontinuous at c
(e.g. $f'(c)$ DNE))

Remark: $f'(c)$ not defined at endpoints $\Rightarrow$ Endpoints are critical points

First Derivative Test for Relative Extrema

Assume c is a critical number of f and f continuous at c. Then

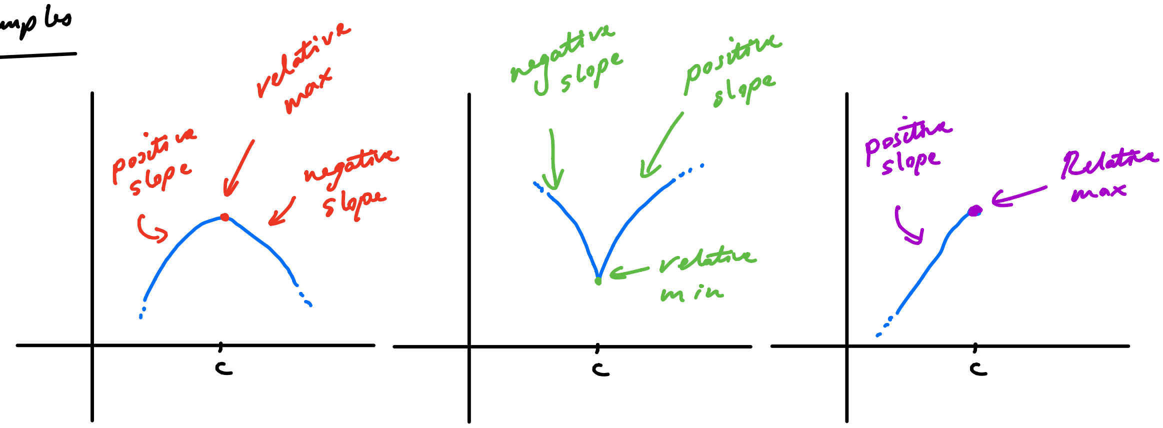
+ c −
$f'(x) > 0$ | $f'(x) < 0$ $\Rightarrow$ $f(c)$ a relative max

− c +
$f'(x) < 0$ | $f'(x) > 0$ $\Rightarrow$ $f(c)$ a relative min

Remark We can also use this at endpoints:

- $+$ on the left of c ($f'(x) > 0$), $f'(x)$ DNE on the right $\Rightarrow$ $f(c)$ a relative max
- $-$ on the left of c ($f'(x) < 0$), $f'(x)$ DNE on the right $\Rightarrow$ $f(c)$ a relative min
- $f'(x)$ DNE on the left of c, $-$ on the right ($f'(x) < 0$) $\Rightarrow$ $f(c)$ a relative max
- $f'(x)$ DNE on the left of c, $+$ on the right ($f'(x) > 0$) $\Rightarrow$ $f(c)$ a relative min

Examples

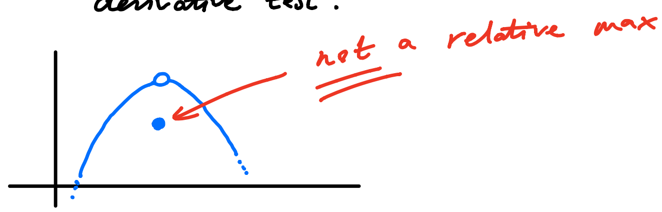


Strategy to find Relative Extrema of f

1. Calculate f'
2. Do sign analysis on f'
3. Apply First Derivative test at critical numbers where f continuous

WARNING: If f <u>not</u> continuous at c cannot apply first derivative test.

E.g.



Examples

Find relative extrema of $f(x) = 2x^3 - 3x^2 - 72x + 15$.

1/ $f'(x) = 6x^2 - 6x - 72 = 6(x^2 - x - 12) = 6(x-4)(x+3)$

2/ A/ $f'(x) = 0 \Rightarrow 6(x-4)(x+3) = 0 \Rightarrow x = -3, 4$

B/ f' a polynomial $\Rightarrow$ continuous everywhere.

$\Rightarrow$ $-3, 4$ are only critical numbers

$f'(x)$

+	-3	$-$	4	+
$f'(-4) = 6\cdot(-8)\cdot(-1)$		$f'(0) = 6\cdot(-4)\cdot 3 < 0$		$f'(5) = 6\cdot 1 \cdot 8 > 0$

3/ f polynomial $\Rightarrow$ continuous at -3 and 4

$\Rightarrow$ $f(-3)$ relative max

$f(4)$ relative min

Example

Find relative extrema of $f(x) = x^3 e^x$.

1/ $f(x) = u(x)v(x)$, $u(x) = x^3$, $v(x) = e^x$

$\Rightarrow u'(x) = 3x^2$, $v'(x) = e^x$

$\Rightarrow f'(x) = u'(x)v(x) + u(x)v'(x) = 3x^2 e^x + x^3 e^x = (3x^2 + x^3)e^x$

$e^x \neq 0$

2/ A/ $f'(x) = 0 \Rightarrow (3x^2 + x^3)e^x = 0 \Rightarrow 3x^2 + x^3 = 0$

$\Rightarrow x^2(3 + x) = 0 \Rightarrow x = 0$ or -3

B/ f' continuous everywhere.

$f'(x)$

$-$	-3	+	0	+
$f'(-4) = -16e^{-4} < 0$		$f'(-1) = 2e^{-1} > 0$		$f'(1) = 4e > 0$

3/ f continuous everywhere

$\Rightarrow$ The only relative extrema is a relative min at -3.

<u>Example</u>

Find relative extrema at $f(x) = \dfrac{2x^2 - 4x + 3}{x^2 - 2x + 1}$.

1/ $f(x) = \dfrac{u(x)}{v(x)}$, $u(x) = 2x^2 - 4x + 3$, $v(x) = x^2 - 2x + 1 = (x-1)^2$

$\Rightarrow u'(x) = 4x - 4 = 4(x-1)$, $v'(x) = 2x - 2 = 2(x-1)$

$$\Rightarrow f'(x) = \frac{u'(x)\,v(x) - u(x)\,v'(x)}{(v(x))^2} = \frac{4(x-1)(x-1)^2 - (2x^2-4x+3)\cdot 2(x-1)}{(x-1)^4}$$

$$= \frac{4(x-1)^2 - 2(2x^2-4x+3)}{(x-1)^3} = \frac{\cancel{4x^2} - \cancel{8x} + 4 - \cancel{4x^2} + \cancel{8x} - 6}{(x-1)^3}$$

$$= \frac{-2}{(x-1)^3}$$

2/ A/ $f'(x) = 0 \Rightarrow \dfrac{-2}{(x-1)^3} = 0$ (No solutions)

f' rational

B/ f' discontinuous $\Rightarrow$ f' undefined $\Rightarrow (x-1)^3 = 0 \Rightarrow x = 1$

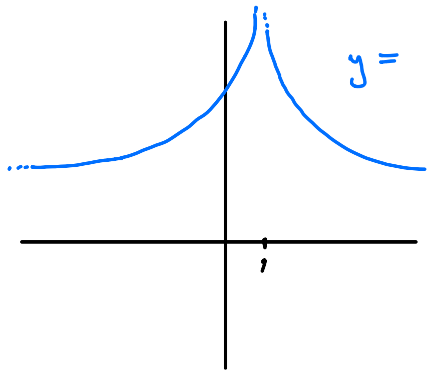
$f'(x)$: $+$ on the left of 1, $-$ on the right of 1

$f'(0) = \dfrac{-2}{(-1)^3} > 0$ $\qquad$ $f'(2) = \dfrac{-2}{(2-1)^3} < 0$

3/ WRONG: f has relative max at 1. $f(1)$ is not even defined. 1 is <u>not</u> a critical number.

There are no critical numbers $\Rightarrow$ There are no relative extrema.

Graph :



$$y = \frac{2x^2 - 4x + 3}{(x-1)^2} = \frac{1}{(x-1)^2} + 2$$

check algebra